Bregman iteration for optical flow

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18th December 2009

Outline

Constrained optimisation problems

- Let J and H be two convex functions from $\mathbb{R}^n$ to $\mathbb{R}$
- Assume $\min_u H(u) = 0$
- Consider the constrained problem

$$\arg\min_u \; J(u) \quad \text{such that} \quad H(u) = 0$$

- Can be very difficult to solve.
  - $H(u) = 0$ can have infinitely many solutions.
  - J and H not necessarily differentiable.

Usual approach

- Constrained optimization problem

$$\arg\min_u \; J(u) \quad \text{such that} \quad H(u) = 0$$

- Approximate solution by solving series of problems

$$\arg\min_u \; J(u) + \lambda_n H(u)$$

for $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_N$.

- Approach is called penalty function/continuation method.

Disadvantages

- Requires very large λ_N for good approximation.
  - Numerically unstable/ill-conditioned for large λ_N.
- Sometimes the λ_i can only be increased in small steps.
  - Algorithm becomes slow due to lots of computations.
- Can be as difficult to solve as initial problem.

Are there alternative approaches without these problems?

- Bregman iteration presents interesting alternative.

Subdifferential and subgradient

Assume $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function.

Definition

We call subdifferential of f at y the set

$$\partial f(y) := \{q \in \mathbb{R}^n : f(x) \geqslant f(y) + \langle q, x - y \rangle, \ \forall x \in \mathbb{R}^n\}$$

Its elements are called subgradients.

Example

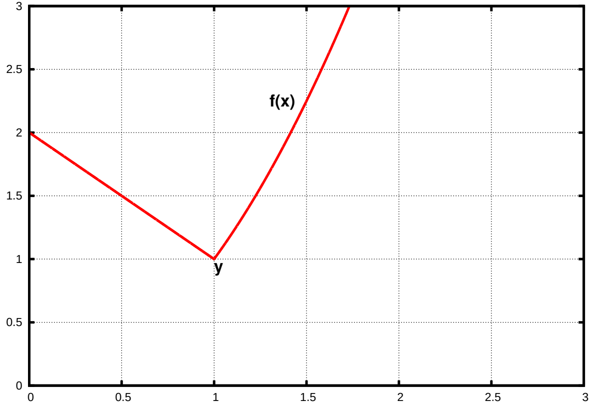


Figure: A convex function f

Example

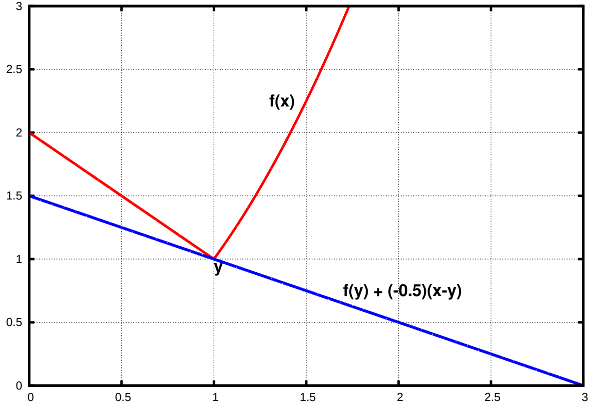


Figure: q$=-\frac{1}{2}$ is a subgradient of f at y

Example

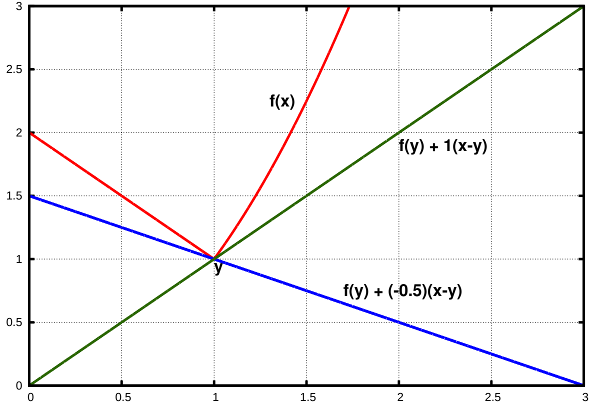


Figure: q=1 another subgradient of f at y

Bregman divergence

Again assume f is a convex function.

Definition

We call Bregman divergence of f the function:

$$D_f^q(x, y) := f(x) - f(y) - \langle q, x - y \rangle$$

- y is an arbitrary but fixed point.
- q is a subgradient of f at y.

Interpretation

- Consider convex and differentiable function $f : \mathbb{R}^n \to \mathbb{R}$
- Linearising f with Taylor expansion around y gives

$$L_{f,y}(x) := f(y) + \langle \nabla f(y), x - y \rangle$$

- Difference between $f(x)$ and $L_{f,y}(x)$ is

$$f(x) - L_{f,y}(x) = f(x) - f(y) - \langle \nabla f(y), x - y \rangle$$

- This is the definition of the Bregman divergence!

Definition of the Bregman iteration

- Bregman iteration is an algorithm for solving

$$\arg\min_u \; J(u) \quad \text{such that} \quad H(u) = 0$$

for convex J and H from $\mathbb{R}^n$ to $\mathbb{R}$ and $\min_u H(u) = 0$

- It computes iteratively

$$\begin{aligned} u^{k+1} &= \arg\min_u \; D_J^{p^k}\left(u, u^k\right) + \lambda H(u) \\ &= \arg\min_u \; J(u) - J\left(u^k\right) - \langle p^k, u - u^k \rangle + \lambda H(u) \\ &= \arg\min_u \; J(u) - \langle p^k, u - u^k \rangle + \lambda H(u) \end{aligned}$$

with $p^k \in \partial J\left(u^k\right)$ and arbitrary $\lambda > 0$.

Advantages of Bregman iteration

At first glance similar to penalty function methods.

However

- $\lambda > 0$ is arbitrary and does not need to be large.
  - Numerically much more stable.
- Only 1 problem to solve instead of series of problems.
  - Faster than penalty function/continuation methods.
- Convergence towards solution of constrained problem.

Convergence results

If $H(u) := \|Au - b\|_2^2$ with matrix A and vector b.

Then one can show

- $0 \leqslant H\left(u^{k+1}\right) \leqslant H\left(u^k\right)$
- If $H\left(u^k\right) = 0$, then u^k also solves

$$\arg\min_u \; J(u) \quad \text{such that} \quad H(u) = 0$$

Preliminary conclusions

- Constrained optimisation problems are difficult to solve.
- Conventional approaches may have disadvantages.
- Bregman iteration is an interesting alternative.

Applying Bregman iteration in optical flow

Simple model:

- Grey value constancy

$$f\,(x+u, y+v, t) = f\,(x, y, t+1)$$

- Linearised constancy assumption.

$$f_x \cdot u + f_y \cdot v + f_t = 0$$

- Flow field should also be smooth (smoothness constraint)

$$\|\nabla u\|_1 + \|\nabla v\|_1 \text{ should be small}$$

Problem formulation

- Find minimizer (u, v) of

$$\arg\min_{u,v} \|\nabla u\|_1 + \|\nabla v\|_1 + \frac{\mu}{2}\|f_x \cdot u + f_y \cdot v + f_t\|_2^2$$

- Difficult to solve: $\|\cdot\|_1$ is not differentiable.

Problem formulation

- Find minimizer (u, v) of

$$\arg\min_{u,v} \|\nabla u\|_1 + \|\nabla v\|_1 + \frac{\mu}{2}\|f_x \cdot u + f_y \cdot v + f_t\|_2^2$$

- Define

$$w := (u, v)^T$$
$$F\, w := f_x \cdot u + f_y \cdot v$$

- Problem rewrites as

$$\arg\min_{w} \|\nabla w\|_1 + \frac{\mu}{2}\|Fw + f_t\|_2^2$$

Problem formulation

Unconstrained problem

$$\arg\min_{w} \|\nabla w\|_1 + \frac{\mu}{2}\|Fw + f_t\|_2^2$$

can be rewritten as constrained problem

$$\arg\min_{w,d} \|d\|_1 + \frac{\mu}{2}\|Fw + f_t\|_2^2 \quad \text{such that} \quad d = \nabla w$$

Problem formulation

- Find minimizer (w, d) of

$$\arg\min_{w,d} \|d\|_1 + \frac{\mu}{2}\|Fw + f_t\|_2^2 \quad \text{such that} \quad d = \nabla w$$

- Define

$$\begin{aligned} \eta &:= (w, d) \\ J(\eta) &:= \|d\|_1 + \frac{\mu}{2}\|Fw + f_t\|_2^2 \\ A(\eta) &:= d - \nabla w \end{aligned}$$

- Problem rewrites as

$$\arg\min_{\eta} \; J(\eta) \quad \text{such that} \quad \|A\eta\|_2^2 = 0$$

Solving through Bregman iteration

- Find minimizer η of

$$\arg\min_{\eta} \; J(\eta) \quad \text{such that} \quad \|A\,\eta\|_2^2 = 0$$

- Apply Bregman iteration:

$$\eta^{k+1} = \arg\min_{\eta} \; D_J^{p^k}\left(\eta, \eta^k\right) + \frac{\lambda}{2}\|A\,\eta\|_2^2$$
$$p^k \in \partial J\left(\eta^k\right)$$

- One can show following simplification is possible:

$$\eta^{k+1} = \arg\min_{\eta} \; J(\eta) + \frac{\lambda}{2}\|A\,\eta - b^k\|_2^2$$
$$b^{k+1} = b^k - A\eta^{k+1}$$

Solving through Bregman iteration

- Leads to this algorithm

$$\underbrace{\left(w^{k+1}, d^{k+1}\right)}_{=\,\eta^{k+1}} = \arg\min_{w,d} \underbrace{\|d\|_1 + \frac{\mu}{2}\|Fw + f_t\|_2^2}_{=\,J(\eta)} + \frac{\lambda}{2}\|\underbrace{d - \nabla w}_{=\,A\eta} - b^k\|_2^2$$

$$b^{k+1} = b^k - \left(d^{k+1} - \nabla w^{k+1}\right)$$

- Easily solvable through simple 3 step algorithm:

Step 1: $w^{k+1} = \arg\min_{w} \frac{\mu}{2}\|Fw + f_t\|_2^2 + \frac{\lambda}{2}\|d^k - \nabla w - b^k\|_2^2$

Step 2: $d^{k+1} = \arg\min_{d} \|d\|_1 + \frac{\lambda}{2}\|d - \nabla w^{k+1} - b^k\|_2^2$

Step 3: $b^{k+1} = b^k - \left(d^{k+1} - \nabla w^{k+1}\right)$

Solving step 1

$$w^{k+1} = \arg\min_{w} \ \frac{\mu}{2}\|Fw + f_t\|_2^2 + \frac{\lambda}{2}\|d^k - \nabla w - b^k\|_2^2$$

- Cost function is differentiable. Requires solving linear system.
  - Matrix is positive definite, sparse and very large.
  - Use Gauss-Seidel or similar algorithms.

Solving step 2

$$d^{k+1} = \arg\min_{d} \|d\|_1 + \frac{\lambda}{2}\|d - \nabla w^{k+1} - b^k\|_2^2$$

- Step 2 has analytical solution:

$$d_j^{k+1} = \text{shrink}\left(\left(\nabla w^{k+1} + b^k\right)_j, \frac{1}{\lambda}\right) \quad j = 1, \dots, n$$

where $\text{shrink}(x, \gamma) := \frac{x}{|x|} \max(|x| - \gamma, 0)$.

  - Operates componentwise.
  - Only simple operations. Very fast.

Solving step 3

$$b^{k+1} = b^k - \left(d^{k+1} - \nabla w^{k+1}\right)$$

- Step 3 is just a simple computation.

Example

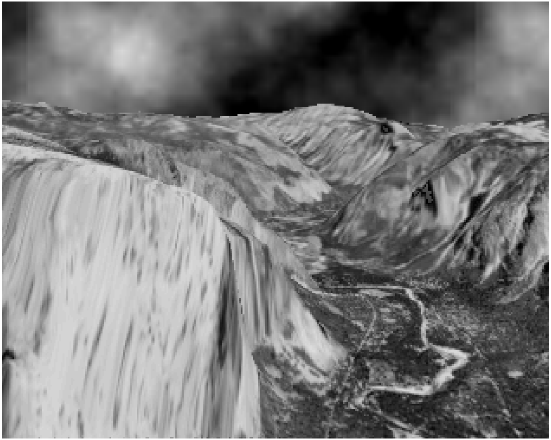


Figure: Yosemite sequence frame 10

Example

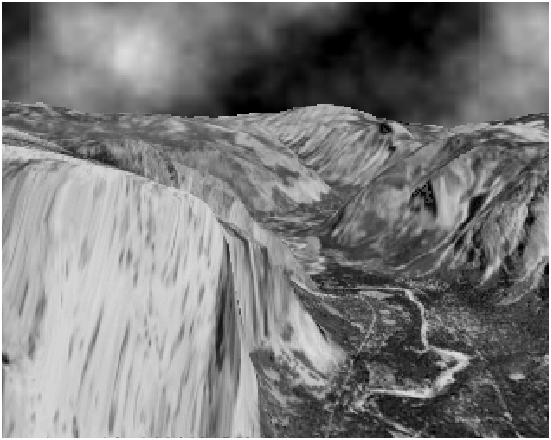
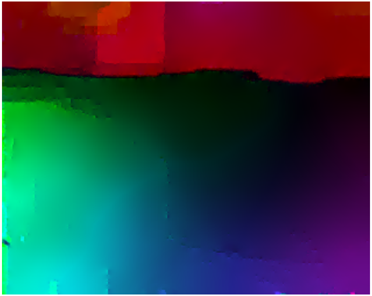
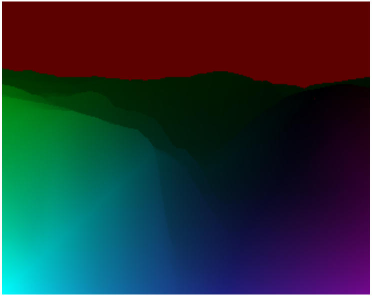


Figure: Yosemite sequence frame 11

Example



(a) Result



(b) Exact solution

Figure: Qualitative comparison: Bregman result and exact ground truth

Conclusion

+ Results look as expected.
  - Average angular error of 5.65.
+ Only 65 Iterations, 5 Iterations for Gauss-Seidel each time.
  - Algorithm is very fast.
+ Very simple implementation.

– 4 different parameters. Not clear how to chose them.
  - In this case: $\lambda = 36.52$, $\mu = 0.49$, 65 and 5 iterations

+ Useful for functionals that are difficult to minimize.

Aims of the thesis

- Find good models for the Bregman Iteration.
- Theoretical and numerical evaluation
  - Convergence behaviour
  - Comparison to other modern approaches

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Thank you for your attention.